Multiple Choice

1 B) $\sqrt{1^2+18^2+6^2}=19$

2 B) $3-i$ is also a root, since all coefficients are real.

$-p=(3+i)+(3-i)=6, \therefore p=-6,$

$q=(3+i)(3-i)=10.$

3 C) $\dfrac{x-1}{-2}=\dfrac{y-3}{4}.$

$\therefore 4x-4=-2y+6.$

$\therefore y+2x=5.$

4 A) Let $z=r\operatorname{cis}\theta,$

$\therefore \dfrac{z^2}{|z|}=\dfrac{r^2\operatorname{cis}2\theta}{r}=r\operatorname{cis}2\theta.$

5 C) Let $x=A\cos nt, \dot{x}=-nA\sin nt, \ddot{x}=-n^2A\cos nt,$

$\therefore \dot{x}_{\max}=nA=6, \ddot{x}_{\max}=n^2A=4.$

$\therefore n=\dfrac{\ddot{x}_{\max}}{\dot{x}_{\max}}=\dfrac{4}{6}=\dfrac{2}{3}.$

$\therefore \text{Period}=\dfrac{2\pi}{n}=3\pi.$

6 A) $x^2+4x+10=(x+2)^2+6.$

7 D) To disprove a proposition is to find a counter-example that the proposition is false.

Here, $2^{11}-1$ is not prime, but 11 is prime.

8 B) The negation of 'if A then B' is 'A and not B'

9 C) Let $e^{i\theta}=\cos\theta+i\sin\theta.$

$\therefore e^{i\theta}\pm 2=\cos\theta\pm 2+i\sin\theta.$

$\therefore \left|e^{i\theta}\pm 2\right|=\sqrt{(\cos\theta\pm 2)^2+\sin^2\theta}=\sqrt{5\pm 4\cos\theta}.$

$\therefore$ The maximum value of $\sqrt{5-4\cos\theta}+\sqrt{5+4\cos\theta}$ is $2\sqrt{5}$, which occurs when $\theta=\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}.$

10 B) $\int_0^{2a} f(x)dx=\int_0^{a} f(x)dx+\int_a^{2a} f(x)dx.$

For $\int_a^{2a} f(x)dx$, let $u=2a-x, du=-dx.$

When $x=2a, u=0$, when $x=a, u=a.$

$$\int_a^{2a} f(x)dx=\int_a^{0} f(2a-u)(-du)$$
$$=\int_0^{a} f(2a-x)dx$$

Question 11

(a) (i) $|w|=\sqrt{1^2+4^2}=\sqrt{17}.$

(ii) $w\bar{z}=(-1+4i)(2+i)$
$=-2-4-i+8i$
$=-6+7i$

(b) Let $u=\ln x, du=\dfrac{1}{x}dx.$ Let $dv=x, v=\dfrac{x^2}{2}.$

$$\int_1^e x\ln x\,dx=\left[\frac{x^2}{2}\ln x\right]_1^e-\int_1^e\frac{x}{2}dx$$
$$=\frac{e^2}{2}-\left[\frac{x^2}{4}\right]_1^e$$
$$=\frac{e^2}{2}-\frac{e^2}{4}+\frac{1}{4}$$
$$=\frac{e^2}{4}+\frac{1}{4}.$$

(c) $a=\dfrac{vdv}{dx}=v^2+v.$

$\therefore \dfrac{dv}{dx}=v+1.$

$\int dx=\int\dfrac{dv}{v+1}.$

$x=\ln(v+1)+C.$

When $x=0, v=1, \therefore C=-\ln 2.$

$\therefore x=\ln\dfrac{v+1}{2}.$

(d) $\underset{\sim}{u}-\underset{\sim}{v}=-(2+p)\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}, \underset{\sim}{u}+\underset{\sim}{v}=(-2+p)\underset{\sim}{i}+5\underset{\sim}{k}.$

If they are perpendicular then $(\underset{\sim}{u}-\underset{\sim}{v}).(\underset{\sim}{u}+\underset{\sim}{v})=0.$

$(\underset{\sim}{u}-\underset{\sim}{v}).(\underset{\sim}{u}+\underset{\sim}{v})=-(2+p)(-2+p)+5$
$=-p^2+4+5$
$=-p^2+9$
$=0.$

$\therefore p=\pm 3.$

(e) $z=\dfrac{-3\pm\sqrt{9-4(3-i)}}{2}$
$=\dfrac{-3\pm\sqrt{-3+4i}}{2}.$

but $\sqrt{-3+4i}=\sqrt{1-4+2\times 1\times 2i}=1+2i.$

$\therefore z=\dfrac{-3\pm(1+2i)}{2}=-1+i$ or $-2-i.$

Question 12

(a) (i) $R + 200\sin 30^\circ = 50g = 500$, since $g = 10$.

$R + 100 = 500$

$R = 400.$

(ii) $\sum f = 200\cos 30^\circ - 0.3R$

$= 100\sqrt{3} - 120$

$\simeq 53.2$ newtons.

(iii) $a = \dfrac{53.2}{50} = 1.064$, since $\sum f = ma = 50a$.

$a = \dfrac{dv}{dt} = 1.064$

$\int_0^V dv = \int_0^3 1.064\,dt$

$V = 1.064t = 1.064 \times 3 = 3.192 \simeq 3.2$ km.

(b) (i) Horizontally, $\ddot{x} = 0$.

$\dot{x} = \int (0)\,dt = C.$

When $t = 0, \dot{x} = u\cos\theta, \therefore C = u\cos\theta$.

$\therefore \dot{x} = u\cos\theta.$

$x = \int (u\cos\theta)\,dt = ut\cos\theta + C.$

When $t = 0, x = 0, \therefore C = 0.$

$\therefore x = ut\cos\theta.$ (1)

Vertically, $\ddot{y} = -g$.

$\dot{y} = \int (-g)\,dt = -gt + C.$

When $t = 0, \dot{y} = u\sin\theta, \therefore C = u\sin\theta$.

$\therefore \dot{y} = -gt + u\sin\theta.$

$y = \int (-gt + u\sin\theta)\,dt = -\dfrac{1}{2}gt^2 + ut\sin\theta + C.$

When $t = 0, y = 0, \therefore C = 0.$

$\therefore y = -\dfrac{1}{2}gt^2 + ut\sin\theta.$ (2)

(ii) From (1), $t = \dfrac{x}{u\cos\theta}$

Substituting to (1),

$$y = -\frac{gx^2}{2u^2\cos^2\theta} + x\tan\theta$$

$$= -\frac{gx^2}{2u^2}\left(\sec^2\theta - \frac{2u^2}{gx}\tan\theta\right)$$

$$= -\frac{gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right)$$

(iii) When $y = 0, x = R, \therefore \tan^2\theta - \dfrac{2u^2}{gR}\tan\theta + 1 = 0.$

This equation has 2 real and distinct values of $\tan\theta$, hence,

2 distinct values of θ when $\Delta = \dfrac{4u^4}{g^2R^2} - 4 > 0.$

$\therefore u^4 > g^2R^2.$

$\therefore u^2 > gR.$

Question 13

(a) Period $= \dfrac{\pi}{3} = \dfrac{2\pi}{n}, \therefore n = 6.$

Let $x = \sqrt{3} + A\cos 6t$.

When $t = 0, x = 2\sqrt{3}, \therefore A = \sqrt{3}.$

$\therefore x = \sqrt{3}(1 + \cos 6t).$

(b) The 2 lines intersect if there is a point that belongs to both lines.

$3 + \lambda_1 = 3 - 2\lambda_2$ (1)

$-1 + 2\lambda_1 = -6 + \lambda_2$ (2)

$7 + \lambda_1 = 2 + 3\lambda_2$ (3)

$(1) - (3)$ gives $-4 = 1 - 5\lambda_2, \therefore \lambda_2 = 1.$

Substituting $\lambda_2 = 1$ to (1) gives $\lambda_1 = -2.$

Test: Substituting $\lambda_2 = 1$ to (2) gives $\lambda_1 = -2$, too.

$\therefore$ The point of intersection is $\begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$.

(c) (i) $(a+b)^2 = (a-b)^2 + x^2$, by Pythagoras' theorem,

$\therefore a^2 + b^2 + 2ab = a^2 + b^2 - 2ab + x^2.$

$\therefore x^2 = 4ab.$

$\therefore x = 2\sqrt{ab}.$

$a + b$ is the hypotenuse, $\therefore a + b \ge 2\sqrt{ab}, \therefore \dfrac{a+b}{2} \ge \sqrt{ab}.$

(ii) Let $a = p^2, b = 4q^2$,

From the result of part (i),

$p^2 + 4q^2 \ge 2\sqrt{4p^2q^2} = 4pq.$

(d) (i) Let $e^{in\theta} = \cos n\theta + i\sin n\theta$,

$\therefore e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$, since $\cos x$ is even, and $\sin x$ is odd.

$\therefore e^{in\theta} + e^{-in\theta} = 2\cos n\theta.$

(ii) $\left(e^{i\theta} + e^{-i\theta}\right)^4 = e^{i4\theta} + 4e^{i3\theta}e^{-i\theta} + 6e^{i2\theta}e^{-i2\theta} + 4e^{i\theta}e^{-i3\theta} + e^{-i4\theta}$

$= e^{i4\theta} + e^{-i4\theta} + 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6$

$\therefore (2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6.$

$\therefore \cos^4\theta = \dfrac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).$

(iii) $\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta = \dfrac{1}{8}\int_0^{\frac{\pi}{2}} (\cos 4\theta + 4\cos 2\theta + 3)\,d\theta$

$= \dfrac{1}{8}\left[\dfrac{1}{4}\sin 4\theta + \dfrac{4}{2}\sin 2\theta + 3\theta\right]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{8} \times \dfrac{3\pi}{2} = \dfrac{3\pi}{16}.$

Question 14

(a) (i) Let $z_1 = re^{i\theta}, \therefore z_2 = re^{i\left(\theta+\frac{\pi}{3}\right)}$.

Point B has the same magnitude as point A and is inclined an angle of $\left(\theta+\frac{\pi}{3}\right), \therefore OAB$ is an equilateral triangle.

(ii) $z_2 = e^{\frac{i\pi}{3}} z_1$.

$$z_2^{\,3} = e^{i\pi} z_1^{\,3}$$
$$= -z_1^{\,3}.$$
$$\therefore z_1^{\,3} + z_2^{\,3} = 0.$$
$$\therefore (z_1+z_2)\left(z_1^{\,2}+z_2^{\,2}-z_1z_2\right)=0.$$
$$\therefore z_1^{\,2}+z_2^{\,2} = z_1z_2, \text{ since } z_1+z_2 \neq 0.$$

(b) $a = \frac{dv}{dt} = 10\left(1-(0.01v)^2\right)$

$$= 10(1-0.01v)(1+0.01v)$$
$$\int_0^V \frac{dv}{(1-0.01v)(1+0.01v)} = 10\int_0^5 dt$$
$$\frac{1}{2}\int_0^V \left(\frac{1}{1-0.01v}+\frac{1}{1+0.01v}\right)dv = 50.$$
$$\frac{1}{2}\times\frac{1}{0.01}\left[\ln\frac{1+0.01v}{1-0.01v}\right]_0^V = 50$$
$$\ln\frac{1+0.01v}{1-0.01v} = 1.$$
$$\frac{1+0.01v}{1-0.01v} = e$$
$$(1+e)0.01v = e-1.$$
$$\therefore v = \frac{100(e-1)}{e+1}.$$

(c) Let $n=2$, $\text{LHS} = \frac{1}{4}$, $\text{RHS} = \frac{1}{2}$.

$\frac{1}{4} < \frac{1}{2}, \therefore$ It's true for $n=2$.

Assume $\exists n \in N : \frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} < \frac{n-1}{n}$.

RTP $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\frac{1}{(n+1)^2} < \frac{n}{n+1}$.

$$\text{LHS} < \frac{n-1}{n}+\frac{1}{(n+1)^2}$$
$$= \frac{(n-1)(n+1)^2+n}{n(n+1)^2}$$
$$= \frac{(n^2-1)(n+1)+n}{n(n+1)^2}$$
$$= \frac{n^3+n^2-1}{n(n+1)^2}$$
$$= \frac{n^2(n+1)}{n(n+1)^2} - \frac{1}{n(n+1)^2}$$
$$= \frac{n}{n+1} - \frac{1}{n(n+1)^2}$$
$$< \frac{n}{n+1} = \text{RHS}.$$

$\therefore$ By the principle of Induction, it is true for $\forall n \geq 2$.

(d) Assume that $\log_n(n+1)$ is rational, let

$\log_n(n+1) = \frac{p}{q}$, where p and q are positive integers.

$$\therefore n+1 = n^{\frac{p}{q}}$$
$$\therefore (n+1)^q = n^p.$$

Assume that n is even, then LHS is odd while RHS is even, it's absurd. The argument is similar if n is odd.

$\therefore \log_n(n+1)$ is irrational.

Question 15

(a) (i) $k^3+1=(k+1)(k^2-k+1)$.

$\therefore$ If $k+1$ is divisible by 3 then k^3+1 is divisible by 3.

(ii) If k^3+1 is not divisible by 3 then $k+1$ is not divisible by 3.

(iii) The converse is: If k^3+1 is divisible by 3 then $k+1$ is divisible by 3.

Since k is an integer, let $p\in Z$, consider these cases:

If $k=3p$, then $k^3+1=(3p)^3+1$ is not divisible by 3.

If $k=3p-1$, $k^3+1=(3p)\left((3p-1)^2-(3p-1)+1\right)$,

which clearly is divisible by 3.

If $k=3p+1$, $k^3+1=(3p+2)\left((3p+1)^2-(3p+1)+1\right)$

$=(3p+2)\left(9p^2+3p+1\right)$, which is not divisible by 3.

$\therefore$ k^3+1 is divisible by 3 only if $k=3p-1$, i.e. $k+1$ is divisible by 3.

(b) (i) Given $\overrightarrow{CB}=\frac{m}{n}\overrightarrow{AC}, \therefore \underset{\sim}{b}-\underset{\sim}{c}=\frac{m}{n}(\underset{\sim}{c}-\underset{\sim}{a})$ then

$$\underset{\sim}{c}\left(1+\frac{m}{n}\right)=\underset{\sim}{b}+\frac{m}{n}\underset{\sim}{a}.$$

$$\therefore \underset{\sim}{c}=\frac{\underset{\sim}{b}+\frac{m}{n}\underset{\sim}{a}}{1+\frac{m}{n}}=\frac{n\underset{\sim}{b}+m\underset{\sim}{a}}{m+n}.$$

$$\therefore \overrightarrow{AC}=\underset{\sim}{c}-\underset{\sim}{a}=\frac{n\underset{\sim}{b}+m\underset{\sim}{a}}{m+n}-\underset{\sim}{a}=\frac{n\underset{\sim}{b}-n\underset{\sim}{a}}{m+n}=\frac{n(\underset{\sim}{b}-\underset{\sim}{a})}{m+n}.$$

(ii) Proven above.

(iii) $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{OR}=\underset{\sim}{p}+\underset{\sim}{r}$.

From (i), $\overrightarrow{OS}=\frac{1}{2}\left(\underset{\sim}{p}+\underset{\sim}{r}\right)+\frac{1}{2}\underset{\sim}{r}=\frac{1}{2}\underset{\sim}{p}+\underset{\sim}{r}$.

$\overrightarrow{PR}=\underset{\sim}{r}-\underset{\sim}{p}$.

Let $OT=mOS$ and $PT=nPR$,

$$\overrightarrow{OT}=m\left(\frac{1}{2}\underset{\sim}{p}+\underset{\sim}{r}\right) \quad (1)$$

and $\overrightarrow{PT}=n\left(\underset{\sim}{r}-\underset{\sim}{p}\right)$. (2)

In $\Delta OPT, \overrightarrow{OT}=\overrightarrow{OP}+\overrightarrow{PT}=\underset{\sim}{p}+n\left(\underset{\sim}{r}-\underset{\sim}{p}\right)$

$=(1-n)\underset{\sim}{p}+n\underset{\sim}{r}$. (3)

Equating the coefficients of $\underset{\sim}{p}$ and $\underset{\sim}{r}$ in (1) and (3) gives

$\frac{m}{2}=1-n$ and $m=n$.

$\therefore m=n=\frac{2}{3}$.

(1) becomes $\overrightarrow{OT}=\frac{1}{3}\underset{\sim}{p}+\frac{2}{3}\underset{\sim}{r}$.

(iv) (2) becomes $\overrightarrow{PT}=\frac{2}{3}\left(\underset{\sim}{r}-\underset{\sim}{p}\right)=\frac{2}{3}\overrightarrow{PR}, \therefore P$ divides PR in the ratio of 2:1.

Question 16

(a) (i) Let T be the tension of the string.

Resolving the forces,

at the $2m$ mass, $2ma=-2mg-kv+T$. (1)

at the $4m$ mass, $4ma=4mg-kv-T$. (2)

$(1)+(2)$ becomes

$6ma=2mg-2kv$.

$$\therefore a=\frac{dv}{dt}=\frac{gm-kv}{3m}.$$

(ii) $$\int\frac{dv}{gm-kv}=\frac{1}{3m}\int dt.$$

$$-\frac{1}{k}\ln(gm-kv)=\frac{t}{3m}+C.$$

When $t=0, v=0, \therefore C=-\frac{1}{k}\ln(gm)$.

$$\frac{1}{k}\ln\frac{gm}{gm-kv}=\frac{t}{3m}.$$

When $t=\frac{3m}{k}\ln 2, \ln\frac{gm}{gm-kv}=\ln 2$.

$$\frac{gm}{gm-kv}=2.$$

$gm=2gm-2kv$.

$$\therefore v=\frac{gm}{2k}.$$

(b) (i) Using IBP, let $u=\sin^{2n}(2\theta), dv=\sin(2\theta)d\theta$.

$du=4n\sin^{2n-1}(2\theta)\cos(2\theta)d\theta, v=-\frac{1}{2}\cos(2\theta)$.

$$I_n=\left[-\frac{1}{2}\sin^{2n}(2\theta)\cos 2\theta\right]_0^{\frac{\pi}{2}}+2n\int_0^{\frac{\pi}{2}}\sin^{2n-1}(2\theta)\cos^2 2\theta d\theta$$

$$=2n\int_0^{\frac{\pi}{2}}\sin^{2n-1}(2\theta)\left(1-\sin^2 2\theta\right)d\theta$$

$=2nI_{n-1}-2nI_n$.

$$\therefore I_n=\frac{2n}{2n+1}I_{n-1}.$$

(ii) $$I_n=\frac{2n}{2n+1}I_{n-1}.$$

$$I_{n-1}=\frac{2(n-1)}{2n-1}I_{n-2}.$$

...

$$I_2=\frac{4}{5}I_1.$$

$$I_1=\frac{2}{3}I_0.$$

$$I_0=\int_0^{\frac{\pi}{2}}\sin 2\theta d\theta=-\left[\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$=-\left(-\frac{1}{2}-\frac{1}{2}\right)=1.$$

$\therefore I_n = \frac{2n}{2n+1}.\frac{2(n-1)}{2n-1}...\frac{4}{5}.\frac{2}{3}$

$= \frac{2n}{2n+1}.\frac{2n}{2n}\frac{2(n-1)}{2n-1}...\frac{4}{5}.\frac{4}{4}.\frac{2}{3}.\frac{2}{2}$

$= \frac{\left(2^n n!\right)^2}{(2n+1)!}$

$= \frac{2^{2n}(n!)^2}{(2n+1)!}.$

(iii) Let $x = \sin^2\theta, dx = 2\cos\theta\sin\theta d\theta.$

When $x = 0, \theta = 0.$ When $x = 1, \theta = \frac{\pi}{2}.$

$\int_0^1 x^n(1-x)^n\,dx = 2\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n}\theta\cos\theta\sin\theta d\theta$

$= \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n} 2\theta\sin 2\theta d\theta$

$= \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1} 2\theta d\theta$

$= \frac{1}{2^{2n}} I_n.$

$\therefore J_n = \frac{(n!)^2}{(2n+1)!}$, using the result above.

(iv) $I_0 = 1$, from above.

$\therefore I_n \le 1$ for all $n = 0,1,2,...,$ since for $0 < x < \frac{\pi}{2}$,
$0 < \sin(2\theta) < 1, \therefore$ the higher the power, the smaller the value.

$\therefore I_n \le 1$

$\therefore \frac{2^{2n}(n!)^2}{(2n+1)!} \le 1$

$\therefore \left(2^n(n!)\right)^2 \le (2n+1)!$